One hundred years of PVI, the Fuchs–Painlevé equation

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PREFACE

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The nonlinear ordinary differential equation

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2}+\left[\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right]\frac{\mathrm{d}\lambda}{\mathrm{d}t}-\frac{1}{2}\left[\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right]\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2$$
$$=\frac{1}{2}\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[k_\infty-k_0\frac{t}{\lambda^2}+k_1\frac{t-1}{(\lambda-1)^2}-(k_t-1)\frac{t(t-1)}{(\lambda-t)^2}\right] \tag{1}$$

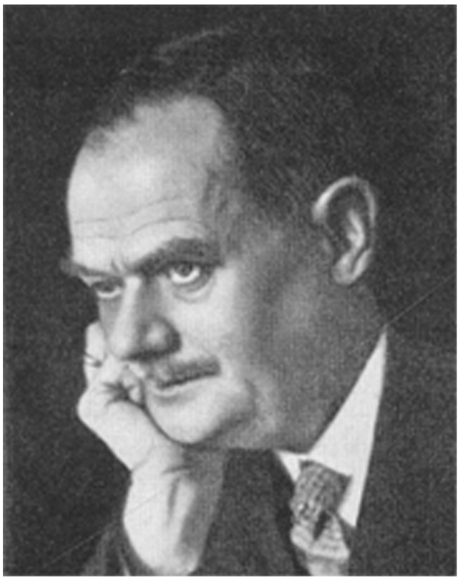
which is nowadays known as the *Painlevé VI (PVI) equation*, is one of the most important differential equations in mathematical physics. It was discovered just over 100 years ago by Richard Fuchs (son of the famous mathematician Immanuel Lazarus Fuchs) and reported for the first time as a paper in *Comptes Rendus de l'Académie des Sciences Paris* 1905 **141** 555–558 (a copy of which follows at the end of this preface). A year later B Gambier, in his seminal paper of 1906, included this equation as the top equation in the list of what are now known as the six Painlevé transcendental equations. The Painlevé list emerged from the work on the classification of all ordinary-second order differential equations whose general solutions are 'uniform', in the sense that there are no movable (i.e. as a function of the initial data) singularities other than poles. This property is known as the Painlevé property. The six equations (in addition to a few equivalent forms) are the only second-order ODEs (of first degree in the leading derivative) whose general solution for generic parameters cannot be expressed in terms of any known elementary or special functions, i.e. they are truly transcendental and irreducible (the full proof of irreducibility was given only in the last decade by K Nishioka and H Umemura).

The way in which Richard Fuchs arrived at the equation, which really should be called the Fuchs–Painlevé equation, was in itself a signifcant step. Following the work of his father (published in the *Sitzungberichte der Berliner Akademie der Wissenschaften* 1888–1898) he looked for second-order *linear* differential equations with four essential singularities (without loss of generality taken at $0, 1, \infty$ and t) such that the 'coefficients of substitution' of the fundamental system of solutions, when circulating the independent variable x around the singularities, are independent of the movable singularity t. The linear differential equation was prescribed to be of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left[\frac{a}{x^2}+\frac{b}{(x-1)^2}+\frac{c}{(x-t)^2}+\frac{\alpha}{x}+\frac{\beta}{x-1}+\frac{\gamma}{x-t}+\frac{e}{(x-\lambda)^2}+\frac{\varepsilon}{x-\lambda}\right]y \tag{2}$$

and the above condition, which in modern language would be called the property that the monodromy of the equation is preserved while varying t, leads to the condition that one can supply an additional linear equation of the form

$$\frac{\partial y_i}{\partial t}=By_i+A\frac{\partial y_i}{\partial x} \tag{3}$$



Richard Fuchs (1873–1945). (Reproduced from the Technischen Universität Berlin.)

on the independent solutions y_i, $i = 1, 2$ of the differential equation (2), as was predicted by general considerations of Fuchs' father (cf L Fuchs, in *Sitzung-Berichte der Berliner Akademie* 1893 and 1894). Notably in equation (2) the 'apparent' singularity λ (i.e. a singularity for which the roots of the characteristic equation for the corresponding series expansion differ by an integer, without the appearance, however, of logarithmic terms in the corresponding series solutions), which should be present according to a result of Poincaré, will in principle depend on the movable singularity t, and as a function of t, $\lambda = \lambda(t)$ was shown by Fuchs to solve the nonlinear ordinary differential equation (1). Nowadays, the overdetermined system consisting of both (2) and (3) would be called a *Lax pair*, in the modern language of soliton theory, or more precisely an *isomonodromic deformation problem*. Thus, Fuchs, had already in his paper of 1905, introduced a number of concepts which attained a deep significance about seventy years later, in particular with the work of K Okamoto in the 1970s, by H Flaschka and A Newell and notably the series of works by M Jimbo, T Miwa and K Ueno in 1980 where the isomonodromic theory of the Painlevé equations was developed further.

Another remarkable observation in Fuchs' 1905 paper, and which was further developed in his subsequent longer paper of 1907 (which is more often cited in the modern literature), is that by introducing the incomplete elliptic integral

$$u = \int_0^\lambda \frac{\mathrm{d}\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}} \tag{4}$$

where notably t enters as the branch point (or modulus) of the elliptic curve, the equation for λ, (1), can be expressed in terms of the Legendre operator for the Picard–Fuchs equation for the corresponding curve. A year later this observation was taken a stage further by Painlevé in his paper in the December issue of the *Comptes Rendus* of 1906 which establishes the elliptic form of the Painlevé VI equation. This latter form plays an important role in the investigation of algebraic solutions of PVI, for special parameter values, and was rediscovered by Yu Manin in 1995 in his study of the mirror symmetries of the projective line. Thus, Fuchs' 1905 paper triggered a number of developments which have come to full fruition only in recent years.

Among the Painlevé equations PVI is the richest because the other equations can be obtained by coalescence on the parameters. All Painlevé equations have been the subject of intensive investigation in the last three decades, and the interest in these equations was revived in the 1970s after it was discovered that forms of these equations emerge in physical problems, e.g. the correlation functions of the 2D Ising model, in quantum spin models, in soliton systems, in quantum gravity, string theory and random matrix theory. In particular PVI has emerged as the key equation in the description of Frobenius manifolds, in connection with particular solutions of the WDVV equations of conformal field theory and in connection with the Ernst reduction of the Einstein equations. It is of historical significance that the year 2005, which celebrated Einstein's remarkable discoveries, coincided with the centenary of the discovery of PVI, and hence this special issue forms a particularly opportune occasion to celebrate this important ordinary differential equation.

The importance of PVI can be recognized by appreciating that this is a universal differential equation, which is the most general (in terms of number of free parameters) of the known equations defining nonlinear special functions. As such, parallels can be drawn between the role played by PVI transcendents in the nonlinear case and the hypergeometric functions at the linear level. In fact, the monograph 'From Gauss to Painlevé' by K Iwasaki, H Kimura, S Shimomura and M Yoshida (Vieweg 1991) draws very clearly the line stretching over more than 150 years of special function theory in which PVI is placed as the key equation. In recent years these lines have been extended into the discrete domain, i.e. the field of nonlinear ordinary *difference* equations, and discrete analogues of PVI have been found which have opened entirely new fields of investigation.

Remarkably, however, in spite of its importance, many open problems exist in the understanding of PVI and of its solutions. This special issue, celebrating the centenary of the discovery of the Painlevé VI equation in 1905, is dedicated to this remarkable equation and its various ramifications. One such ramification is the generalization of equation (1) to higher-order equations which had already been established in 1912 by one of Painlevé's students, René Garnier. The latter adopted Fuchs' original idea of the isomonodromic deformation of a linear second-order differential equation to the case of more than four essential singularities. This leads to a complicated system of partial differential equations in what is now a collection of moving singularities $t_1, \ldots, t_n$, instead of a single variable t, but singling out any particular moving singularity, t_1 say, this *Garnier system* contains in particular a coupled set of second order ODEs with t_1 an independent variable, in terms of which the Painlevé property can be established. We can thus consider this sytem as constituting a hierarchy of higher-order analogues of the Painlevé VI equation.

Another, more recent, development has been the transition from differential to difference equations. In recent years a number of discrete analogues of the Painlevé equations have been discovered, which are ordinary nonlinear second-order difference equations sharing many of the properties (such as the existence of isomonodromic deformation problems) with the original Painlevé differential equations. In particular some discrete analogues of the Painlevé VI equation, which under a specific continuum limit reduce to equation (1), have been proposed by M Jimbo and H Sakai, B Grammaticos and A Ramani, and others. The establishment in 1999 by Sakai of an elliptic discrete Painlevé type equation on the basis of the study of rational surfaces arising from the resolution of singularities of birational mappings, has been one of the major triumphs of this development in the direction of difference equations.

We have tried in this special issue to bring together a number of strands of the modern research on the Painlevé VI equations and its generalizations in the direction of higher-order and discrete systems. Obviously, the collection of research articles in this volume can only represent a snapshot of the ongoing activities in this field. The special issue contains a number of review papers in addition to contributed research papers, which we have divided into a number of topics. The first part, *Basic Theory*, contains a number of topical reviews highlighting the analysis of solutions (review papers by D Guzzetti and A Kitaev) and classification problems (review by C Cosgrove) as well as contributions explaining the hierarchy structure of PVI (by K Fuji and T Suzuki) and the connection with isomonodromic deformation problems on the torus (by Yu Chernyakov *et al*). The second part, *Coalescences and Reductions*, contains contributions highlighting how PVI emerges from reductions of soliton-type systems (by R Conte, A M Grundland and M Musette) as well as the coalescence structure of the Painlevé equations and Garnier systems (contributions by T Suzuki and by Y Ohyama and S Okumura). The third part, *Garnier Systems and Discrete Analogues*, deals with generalizations to higher-order differential equations (S Shimomura) and to discrete analogues of PVI (A Ramani *et al*) and its generalizations (H Sakai) and hierarchies (by S Kakei and T Kikuchi, and by A Tongas and F Nijhoff). The final part, *Applications to Physics*, contains a review on the emergence of PVI in connection with random matrix theory (by P Forrester and N Witte) as well as contributions highlighting connections with statistical mechanics (by V Bazhanov and V Mangazeev, and by S Boukraa *et al*). The role of PVI in connection with general relativity is highlighted in the paper by M Shah and N Woodhouse.

We look forward to the next 100 years of developments on equations related to PVI. Perhaps in that time as much will be known about the elliptic discrete Painlevé type equations as we know now about PVI. Eventually the nonlinear special functions that solve these equations should be incorporated in our mathematical toolkit, as are the well-known linear and autonomous special functions that we all learn about in our basic mathematical education.

P A Clarkson (University of Kent)
N Joshi (University of Sydney)
M Mazzocco (University of Manchester)
F W Nijhoff (University of Leeds)
M Noumi (Kobe University, Japan)

Guest Editors

Ainsi le minimum s'est produit 19 minutes environ après la plus grande phase

CORRESPONDANCE.

M. le **Ministre de l'Instruction publique, des Beaux-Arts et des Cultes** communique à l'Académie deux Rapports transmis par le Vice-Consulat de France, à Messine, à M. le Ministre des Affaires étrangères, relativement aux récents tremblements de terre ressentis en Sicile et en Calabre.

(Renvoi à la Commission de Sismologie.)

M. le **Secrétaire perpétuel** signale, parmi les pièces imprimées de la Correspondance, l'Ouvrage suivant :

L'évolution de la matière, par M. le Dr Gustave Le Bon. (Présenté par M. H. Poincaré.)

ANALYSE MATHÉMATIQUE. — *Sur quelques équations différentielles linéaires du second ordre*. Note de M. **Richard Fuchs**, présentée par M. Poincaré.

Les recherches suivantes se fondent sur les travaux de mon père, L. Fuchs, publiés dans les *Sitzungsberichte der Berliner Akademie der Wissenschaften*, 1888-1898 : *Ueber lineare Differentialgleichungen, deren Substitutionsgruppe von einem in den Coefficienten auftretenden Parameter unabhängig ist.*

Je me propose de chercher la forme d'une équation différentielle linéaire du second ordre telle que les points singuliers essentiels soient 0, 1, t, ∞ et que les coefficients des substitutions, qu'un système fondamental d'intégrales y_1, y_2 subit avec les circulations de la variable x, soient arbitraires et indépendants de t. La connexion de ce problème avec le problème de Riemann (voir les travaux de M. Schlesinger, *Journal de Crelle*, t. 123, 124, 130) est évident.

Suivant les résultats de M. Poincaré obtenus dans les *Acta Mathematica*, t. IV, p. 217-219, il faut que, outre les points singuliers essentiels, il existe un point singulier non essentiel, que je désigne par λ.

Soit ainsi l'équation

$$(1)\quad \left\{\begin{aligned} \frac{d^2y}{dx^2} = \Big[& \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} \\ & + \frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-t} + \frac{e}{(x-\lambda)^2} + \frac{\varepsilon}{x-\lambda}\Big] y. \end{aligned}\right.$$

a, b, c, e sont indépendants de x et de t; α, β, γ, ε sont des fonctions de t indépendantes de x. D'après la supposition faite, il existe ([1]) une équation

$$(2)\qquad \frac{\partial y_i}{\partial t} = \mathrm{B} y_i + \mathrm{A}\frac{\partial y_i}{\partial x},$$

où B et A sont des fonctions rationnelles de x dont A satisfait ([2]) l'équation

$$(3)\qquad \frac{\partial p}{\partial t} = \mathrm{A}\frac{\partial p}{\partial x} + 2p\frac{\partial \mathrm{A}}{\partial x} - \frac{1}{2}\frac{\partial^3 \mathrm{A}}{\partial x^3},$$

si l'on désigne par p le coefficient de y dans l'équation (1). Pour la fonction A, je trouve la forme

$$(4)\qquad \mathrm{A} = -\frac{x(x-1)}{x-\lambda}\,\frac{t-\lambda}{t(t-1)}.$$

En substituant cette expression dans (3), on obtient des équations de condition qui sont satisfaites, quand la fonction λ de t est intégrale de l'équation

$$(5)\quad \left\{\begin{aligned} & \frac{d^2\lambda}{dt^2} + \left[\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda - t}\right]\frac{d\lambda}{dt} - \frac{1}{2}\left[\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right]\left(\frac{d\lambda}{dt}\right)^2 \\ & \quad = \frac{1}{2}\,\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[k_\infty - k_0\frac{t}{\lambda^2} + k_1\frac{t-1}{(\lambda-1)^2} - (k_t - 1)\frac{t(t-1)}{(\lambda-t)^2}\right], \end{aligned}\right.$$

en posant

$$-4(k-1) + 4a + 4b + 4c = k_\infty \qquad (k = \text{const.}),$$
$$4\left(a + \frac{1}{4}\right) = k_0, \qquad 4\left(b + \frac{1}{4}\right) = k_1, \qquad 4\left(c + \frac{1}{4}\right) = k_t.$$

Les quantités k, a, b, c ou k_0, k_1, k_t, k_∞ restent arbitraires, la quantité e

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([1]) L. Fuchs, *Berliner Sitzungsberichte*, 1888, p. 1278-1282.

([2]) L. Fuchs, *Berliner Sitzungsberichte*, 1894, p. 1124.

reçoit la valeur $\frac{3}{4}$. Les α, β, γ, ε seront des fonctions rationnelles de λ et de $\frac{d\lambda}{dt}$:

$$(6)\quad \begin{cases} \alpha = \dfrac{k\lambda}{t} - \dfrac{\lambda(2\lambda - t - 1)^2}{4t(\lambda-1)(\lambda-t)} - \rho\dfrac{\lambda(\lambda-1)(\lambda-t)}{t} + \dfrac{1}{4}\dfrac{t(t-1)^2}{\lambda(\lambda-1)(\lambda-t)}\left[\dfrac{d\lambda}{dt} - \dfrac{\lambda-1}{t-1}\right]^2, \\ \beta = -\dfrac{k(\lambda-1)}{t-1} + \dfrac{(\lambda-1)(2\lambda-t)^2}{4(t-1)\lambda(\lambda-t)} + \rho\dfrac{\lambda(\lambda-1)(\lambda-t)}{t-1} - \dfrac{1}{4}\dfrac{t^2(t-1)}{\lambda(\lambda-1)(\lambda-t)}\left[\dfrac{d\lambda}{dt} - \dfrac{\lambda}{t}\right]^2, \\ \gamma = \dfrac{k(\lambda-t)}{t(t-1)} - \dfrac{(\lambda-t)(2\lambda-1)^2}{4t(t-1)\lambda(\lambda-1)} - \rho\dfrac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} + \dfrac{1}{4}\dfrac{t(t-1)}{\lambda(\lambda-1)(\lambda-t)}\left[\dfrac{d\lambda}{dt}\right]^2, \\ \varepsilon = -\alpha - \beta - \gamma = -\dfrac{1}{2}\dfrac{2\lambda-1}{\lambda(\lambda-1)} - \dfrac{1}{2}\dfrac{t(t-1)}{\lambda(\lambda-1)(\lambda-t)}\dfrac{d\lambda}{dt}, \\ \rho = \dfrac{a}{\lambda^2} + \dfrac{b}{(\lambda-1)^2} + \dfrac{c}{(\lambda-t)^2}. \end{cases}$$

Le nombre des constantes est justement celui qu'il faut pour déterminer les coefficients des substitutions données pour les points singuliers 0, 1, t.

L'équation (5) en λ peut être mise sous une forme remarquable. En posant

$$(7)\qquad u = \int_0^\lambda \frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}}$$

on a

$$(8)\quad \begin{cases} \dfrac{d^2u}{dt^2} + \dfrac{2t-1}{t(t-1)}\dfrac{du}{dt} + \dfrac{u}{4t(t-1)} \\ \quad = \dfrac{\sqrt{\lambda(\lambda-1)(\lambda-t)}}{2t^2(t-1)^2}\left[k_\infty - k_0\dfrac{t}{\lambda^2} + k_1\dfrac{t-1}{(\lambda-1)^2} - k_t\dfrac{t(t-1)}{(\lambda-t)^2}\right]. \end{cases}$$

Le premier membre est le premier membre de l'équation de Legendre, le second est une fonction doublement périodique de u.

On aura un cas spécial intéressant en choisissant $k_\infty = k_0 = k_1 = k_t = 0$. Dans ce cas, les racines des équations déterminantes fondamentales pour tous les points singuliers essentiels sont égales. Ainsi, on déduira cette équation de l'équation de λ de la même façon qu'on déduit l'équation de Legendre de l'équation de Gauss en prenant les α, β, γ de Gauss de façon que ces racines soient égales.

Le cas spécial donne le résultat suivant : les coefficients des substitutions de y_1, y_2 en circulant les points singuliers pour l'équation

$$(9)\quad \frac{d^2y}{dx^2} + \left[\frac{1}{4x^2} + \frac{1}{4(x-1)^2} + \frac{1}{4(x-t)^2} - \frac{1}{4(x-\lambda)^2} - \frac{\alpha}{x} - \frac{\beta}{x-1} - \frac{\gamma}{x-t} - \frac{\varepsilon}{x-\lambda}\right]y = 0$$

sont indépendants de t, quand u, défini par l'équation

$$u = \int_0^\lambda \frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}},$$

satisfait à l'équation de Legendre :

$$2t(t-1)\frac{d^2u}{dt^2} + 2(2t-1)\frac{du}{dt} + \frac{1}{2}u = 0$$

et qu'on a

$$\begin{aligned}
\alpha &= \frac{1}{4}\left[\frac{1}{\lambda} - \frac{1}{t} - 1\right] + \frac{t(t-1)^2}{4\lambda(\lambda-1)(\lambda-t)}\left[\frac{d\lambda}{dt} - \frac{\lambda-1}{t-1}\right]^2,\\
\beta &= \frac{1}{4}\left[\frac{1}{\lambda-1} - \frac{1}{t-1} + 1\right] - \frac{t^2(t-1)}{4\lambda(\lambda-1)(\lambda-t)}\left[\frac{d\lambda}{dt} - \frac{\lambda}{t}\right]^2,\\
\gamma &= \frac{1}{4}\left[\frac{1}{\lambda-t} + \frac{1}{t} + \frac{1}{t-1}\right] + \frac{t(t-1)}{4\lambda(\lambda-1)(\lambda-t)}\left[\frac{d\lambda}{dt}\right]^2,\\
\varepsilon &= -\alpha - \beta - \gamma = -\frac{1}{2}\left[\frac{1}{\lambda} + \frac{1}{\lambda-1}\right] - \frac{1}{2}\,\frac{t(t-1)}{\lambda(\lambda-1)(\lambda-t)}\,\frac{d\lambda}{dt}.
\end{aligned}$$

ANALYSE MATHÉMATIQUE. — *Sur les surfaces minima*. Note de M. S. **Bernstein**, présentée par M. E. Picard.

L'équation des *surfaces minima* est une des équations du type elliptique qui a le plus attiré l'attention des géomètres du XIX[e] siècle. On n'a cependant pas donné de solution générale et rigoureuse au problème de Plateau ou de Dirichlet. Nous proposons de l'aborder par la *méthode paramétrique* (¹) qui donne cette solution sous forme de série de Mittag-Leffler par rapport à un paramètre α et sous forme d'une série normale par rapport à x et y. Cette méthode nous a montré que *dans la plupart des cas la possibilité d'un problème de Dirichlet était caractérisée par le fait que la solution de l'équation en question ne pouvait avoir de ligne singulière analytique* (nous voulons dire par là que, si une solution existe d'un côté de la courbe analytique $x = f(\theta)$, $y = \varphi(\theta)$, $z = \psi(\theta)$, elle peut être prolongée analytiquement de l'autre côté de cette courbe). En particulier, il est facile de se rendre compte que dans le cas des surfaces minima *le problème de Plateau sera possible si le théorème* A *est exact*.

(¹) *Comptes rendus*, 24 octobre 1904 et 29 mai 1905.